Weighted L^p decay estimates of solutions to the wave equation with a potential

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Abstract

We obtain large time decay estimates on weighted L^p spaces, $2 < p < +\infty$, for solutions to the wave equation with real-valued potential $V(x) = O(\langle x\rangle^{-2-\delta_0})$, $\delta_0 > 0$.

1 Introduction and statement of results

Let $V \in L^\infty(\mathbf{R}^3)$ be a real-valued function satisfying

$$|V(x)| \le C\langle x\rangle^{-2-\delta_0}, \quad \forall x \in \mathbf{R}^3, \tag{1.1}$$

with constants $C > 0$ and $\delta_0 > 0$ not necessarily small, where $\langle x\rangle = (1+|x|^2)^{1/2}$. Denote by G_0 and G the self-adjoint realizations of the operators $-\Delta$ and $-\Delta + V(x)$ on $L^2(\mathbf{R}^3)$. We suppose that G has no eigenvalues, which in turn implies $G \ge 0$. Moreover, under the assumption (1.1), G has no strictly positive resonances (e.g. see [4], [5]).

Given any $a > 0$ denote by $\chi_a \in C^\infty(\mathbf{R})$, $\chi_a \ge 0$, a function supported in the interval $[a, +\infty)$, $\chi_a = 1$ on $[a + 1/2, +\infty)$. It is well known that the solutions to the free wave equation satisfy the following dispersive estimate

$$\left\| G_0^{-\alpha} e^{it\sqrt{G_0}} \right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha}, \quad t \neq 0, \tag{1.2}$$

for every $2 \le p < +\infty$, where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$. Hereafter, given $1 \le p \le +\infty$, L^p denotes the space $L^p(\mathbf{R}^3)$. It turns out that a better decay is possible to get in weighted L^p spaces. Namely, we have the following estimate (see the appendix):

$$\left\| \langle x\rangle^{-\sigma\alpha} G_0^{-\alpha} e^{it\sqrt{G_0}} \chi_a(\sqrt{G_0}) \langle x\rangle^{-\sigma\alpha} \right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha(1+\sigma)}, \quad |t| \ge 1, \tag{1.3}$$

for every $a > 0$, $\sigma \ge 0$, $2 \le p < +\infty$, where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$. The purpose of this note is to prove an analogue of (1.3) for the operator G. Our main result is the following

Theorem 1.1 *Assume (1.1) fulfilled. Then, for every $a > 0$, $2 \le p < +\infty$, $0 < \sigma < \delta_0$, the following estimate holds*

$$\left\| \langle x\rangle^{-\sigma\alpha} G^{-\alpha} e^{it\sqrt{G}} \chi_a(\sqrt{G}) \langle x\rangle^{-\sigma\alpha} \right\|_{L^{p'}\to L^p} \le C\left(|t|^{-1-\sigma}\log(1+|t|)\right)^\alpha, \quad |t| \ge 1, \tag{1.4}$$

where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$, while for $\sigma \ge \delta_0$ we have

$$\left\| \langle x\rangle^{-\sigma\alpha} G^{-\alpha} e^{it\sqrt{G}} \chi_a(\sqrt{G}) \langle x\rangle^{-\sigma\alpha} \right\|_{L^{p'}\to L^p} \le C_\epsilon |t|^{-\alpha(1+\delta_0-\epsilon)}, \quad |t| \ge 1, \quad \forall 0 < \epsilon \ll 1. \tag{1.5}$$

Remark. It follows from (1.5) that for potentials $V(x) = O_N\left(\langle x\rangle^{-N}\right), \forall N \gg 1$, we have the estimate

$$\left\|\psi G^{-\alpha} e^{it\sqrt{G}} \chi_a(\sqrt{G})\psi\right\|_{L^{p'} \to L^p} \le C_N |t|^{-N}, \quad |t| \ge 1, \quad \forall N \gg 1, \tag{1.6}$$

for every $2 < p < +\infty$ and every function $\psi \in C_0^\infty(\mathbf{R}^3)$.

The estimate (1.4) with $\chi_a \equiv 1$, $\sigma = 0$, $2 \le p \le 4$, and without the logarithmic term in the RHS, was proved in [1] for potentials $V(x) = O\left(\langle x\rangle^{-3-\varepsilon_0}\right)$, $\varepsilon_0 > 0$, and later on extended in [4] to non-negative potentials $V(x) = O\left(\langle x\rangle^{-2-\varepsilon_0}\right)$, $\varepsilon_0 > 0$. Recently, in [3] an analogue of (1.4) with $\sigma = 0$ was obtained for a larger class of short-range potentials.

To prove Theorem 1.1 we follow some ideas from [2] and [3]. The proof is based on a carefull study of the operator-valued function

$$\langle x\rangle^{-\sigma}(G - \lambda^2 \pm i0)^{-1}\langle x\rangle^{-\sigma} - \langle x\rangle^{-\sigma}(G_0 - \lambda^2 \pm i0)^{-1}\langle x\rangle^{-\sigma} : L^1 \to L^\infty, \quad \lambda \ge \lambda_0 > 0,$$

together with its derivatives (see Proposition 3.2). This in turn requires sharp estimates for the resolvent of the perturbed operator as well as of its derivatives on weighted L^2 spaces (see Proposition 2.2).

Acknowledgements. A part of this work was carried out while the first author was visiting the University of Nantes in June, 2004, under the support of the agreement Brazil-France in Mathematics - Proc. 69.0014/01-5.

2 Uniform resolvent estimates

Given any $\lambda \ge 0$, $0 < \varepsilon \le 1$, define the free resolvent by

$$R_0(\lambda \pm i\varepsilon) = (G_0 - (\lambda \pm i\varepsilon)^2)^{-1} : L^2 \to L^2,$$

with kernel

$$[R_0(\lambda \pm i\varepsilon)](x, y) = \frac{e^{(\pm i\lambda - \varepsilon)|x-y|}}{4\pi|x-y|}.$$

Then the kernel of $R_0^{(k)} = d^k R_0/d\lambda^k$, $k \ge 1$, is given by

$$[R_0^{(k)}(\lambda \pm i\varepsilon)](x, y) = \frac{(\pm i)^k}{4\pi}|x-y|^{k-1} e^{(\pm i\lambda - \varepsilon)|x-y|}.$$

Proposition 2.1 *Let $s > -1/2$, $s_1 > 1/2$, $0 \le \sigma \le 1$, $\lambda_0 > 0$, and let $k \ge 1$ be an integer. Then the following estimates hold:*

$$\|\langle x\rangle^{-s_1} R_0(\lambda \pm i\varepsilon)\langle x\rangle^{-s_1}\|_{L^2 \to L^2} \le C\lambda^{-1}, \quad \lambda \ge \lambda_0, \tag{2.1}$$

$$\|\langle x\rangle^{-s} R_0(\lambda \pm i\varepsilon)\langle x\rangle^{-s_1}\|_{L^2 \to L^2} \le C\lambda^{-1}\varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \quad \lambda \ge \lambda_0, \tag{2.2}$$

$$\left\|\langle x\rangle^{-k-s} R_0^{(k)}(\lambda \pm i\varepsilon)\langle x\rangle^{-k-s_1}\right\|_{L^2 \to L^2} \le C\lambda^{-1}\varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \quad \lambda \ge \lambda_0, \tag{2.3}$$

$$\|R_0(\lambda \pm i\varepsilon)\langle x\rangle^{-s}\|_{L^2 \to L^\infty} \le C\varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \quad \lambda \ge 0, \tag{2.4}$$

$$\left\|R_0^{(1)}(\lambda \pm i\varepsilon)\langle x\rangle^{-1-s}\right\|_{L^2 \to L^\infty} \le C\varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \quad \lambda \ge 0, \tag{2.5}$$

$$\left\|\langle x\rangle^{-k+1-\sigma}R_0^{(k+1)}(\lambda\pm i\varepsilon)\langle x\rangle^{-k-1-s}\right\|_{L^2\to L^\infty}\le C\varepsilon^{-1+\min\{\sigma,s+1/2-\epsilon\}},\quad \lambda\ge 0, \tag{2.6}$$

$$\left\|\langle x\rangle^{-s}R_0(\lambda\pm i\varepsilon)\right\|_{L^1\to L^2}\le C\varepsilon^{-\max\{1/2-s+\epsilon,0\}},\quad \lambda\ge 0, \tag{2.7}$$

$$\left\|\langle x\rangle^{-1-s}R_0^{(1)}(\lambda\pm i\varepsilon)\right\|_{L^1\to L^2}\le C\varepsilon^{-\max\{1/2-s+\epsilon,0\}},\quad \lambda\ge 0, \tag{2.8}$$

$$\left\|\langle x\rangle^{-k-1-s}R_0^{(k+1)}(\lambda\pm i\varepsilon)\langle x\rangle^{-k+1-\sigma}\right\|_{L^1\to L^2}\le C\varepsilon^{-1+\min\{\sigma,s+1/2-\epsilon\}},\quad \lambda\ge 0, \tag{2.9}$$

$\forall\, 0<\epsilon\ll 1$, *with a constant* $C=C(\epsilon)>0$ *independent of* λ *and* ε, *where the estimates (2.4) and (2.7) hold for* $s\ge 0$ *only. Moreover, the constant* C *in (2.2) and (2.3) may depend on* λ_0.

Proof. The estimate (2.1) is well known and that is why we omit its proof. The estimates (2.2), (2.4), (2.5), (2.7) and (2.8) are proved in [3] for $s\ge 0$. Here we will provide the proof of these estimates (exept for (2.4) and (2.7)) for $-1/2<s<0$. The proof of (2.2) in this case is a little bit more involved, while the proof of (2.5) and (2.8) goes in precisely the same way and we will present it just for the sake of completeness.

To prove (2.2) for this range of values of s we will take advantage of the formula

$$\langle x\rangle^{-s}R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s_1}=R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s-s_1}+R_0(\lambda\pm i\varepsilon)[-\Delta,\langle x\rangle^{-s}]R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s_1}.$$

Taking into account that $[-\Delta,\langle x\rangle^{-s}]=\sum_{j=1}^3 O\left(\langle x\rangle^{-s-1}\right)\partial_{x_j}+O\left(\langle x\rangle^{-s-2}\right)$, we obtain from the above represantation

$$\begin{aligned}\left\|\langle x\rangle^{-s}R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s_1}\right\|_{L^2\to L^2}&\le\left\|R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s-s_1}\right\|_{L^2\to L^2}\\ +C\left\|R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2\to L^2}&\left\|\langle x\rangle^{-s-1/2+\epsilon}\nabla R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s_1}\right\|_{L^2\to L^2}\\ +C\left\|R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2\to L^2}&\left\|\langle x\rangle^{-s-3/2+\epsilon}R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s_1}\right\|_{L^2\to L^2},\end{aligned}$$

for $0<\epsilon\ll 1$. On the other, it is not hard to see that when $s\ge 0$ (2.2) implies

$$\left\|\langle x\rangle^{-s}\nabla R_0(\lambda\pm i\varepsilon)\langle x\rangle^{-s_1}\right\|_{L^2\to L^2}\le C\varepsilon^{-\max\{1/2-s+\epsilon,0\}},\quad \lambda\ge\lambda_0,$$

with a constant $C>0$ depending on λ_0. Now it is easy to see that (2.2) with $-1/2<s<0$ follows from the above estimates and (2.2) with $s\ge 0$.

Let us now see that (2.3) follows from (2.2) by induction in k. Set

$$-\widetilde{\Delta}:=-r\Delta r^{-1}=-\partial_r^2+r^{-2}\Delta_{S^2},$$

where Δ_{S^2} denotes the (positive) Laplace-Beltrami operator on $S^2:=\{x\in\mathbf{R}^3:|x|=1\}$, and denote by $\widetilde{G}_0$ the self-adjoint realization of the operator $-\widetilde{\Delta}$ on the Hilbert space $H=L^2(\mathbf{R}^+\times S^2,drdw)$. Clearly, $\widetilde{G}_0$ is unitary equivalent to G_0, so it suffices to prove (2.3) with G_0 and L^2 replaced by $\widetilde{G}_0$ and H, respectively. Using the identity

$$-2\widetilde{\Delta}+[r\partial_r,-\widetilde{\Delta}]=0, \tag{2.10}$$

we obtain the following representation for the first derivative of the resolvent

$$\widetilde{R}_0(\lambda\pm i\varepsilon)=(\widetilde{G}_0-(\lambda\pm i\varepsilon)^2)^{-1}:H\to H,$$

namely

$$(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s}\widetilde{R}_0^{(1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1} = -\langle r\rangle^{-k-s}\widetilde{R}_0(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1}$$

$$+\langle r\rangle^{-k-s}\widetilde{R}_0(\lambda \pm i\varepsilon)\partial_r r\langle r\rangle^{-k-s_1} - \langle r\rangle^{-k-s}r\partial_r\widetilde{R}_0(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1}.$$

Differentiating this identity $k-1$ times with respect to λ leads to

$$(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s}\widetilde{R}_0^{(k)}(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1} = -2\langle r\rangle^{-k-s}\widetilde{R}_0^{(k-1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1}$$

$$+\langle r\rangle^{-k-s}\widetilde{R}_0^{(k-1)}(\lambda \pm i\varepsilon)\partial_r r\langle r\rangle^{-k-s_1} - \langle r\rangle^{-k-s}r\partial_r\widetilde{R}_0^{(k-1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1}.$$

On the other hand, it is easy to see (for example, this follows from the estimate (2.22) below obtained in a more general situation) that (2.3) with $k-1$ implies

$$\left\|\langle r\rangle^{-k-s}r\partial_r\widetilde{R}_0^{(k-1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-k-s_1}\right\|_{H\to H} \le C\varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \quad \lambda \ge \lambda_0,$$

with a constant $C>0$ depending on λ_0. Therefore, the estimate (2.3) with $k-1$ implies (2.3) with k.

To prove (2.5) for $-1/2 < s < 0$ observe that we have

$$\left\|R_0^{(1)}(\lambda \pm i\varepsilon)\langle x\rangle^{-1-s}\right\|^2_{L^2\to L^\infty} \le \sup_{x\in\mathbf{R}^3} A_s(x,\varepsilon),$$

where

$$A_s(x,\varepsilon) = \int_{\mathbf{R}^3} e^{-2\varepsilon|x-y|}\langle y\rangle^{-2s-2}dy = \int_{|y|\le|x|/2} + \int_{|y|\ge|x|/2}$$

$$\le e^{-\varepsilon|x|}\int_{|y|\le|x|/2}\langle y\rangle^{-2s-2}dy + C\int_{|y|\ge|x|/2} e^{-2\varepsilon|x-y|}\langle x-y\rangle^{-2s-2}dy$$

$$\le Ce^{-\varepsilon|x|}\int_0^{|x|/2}(\rho+1)^{-2s}d\rho + C\int_0^\infty e^{-2\varepsilon\rho}(\rho+1)^{-2s}d\rho$$

$$\le C'e^{-\varepsilon|x|}(|x|+1)^{-2s+1} + C'\int_0^\infty e^{-2\varepsilon\rho}(\rho^{-2s}+1)d\rho \le C''\varepsilon^{-1+2s}.$$

The estimate (2.8) is obtained in precisely the same way.

In what follows we will prove (2.6) and (2.9). We have

$$\left\|\langle x\rangle^{-k+1-\sigma}R_0^{(k+1)}(\lambda \pm i\varepsilon)\langle x\rangle^{-k-1-s}\right\|^2_{L^2\to L^\infty} \le \sup_{x\in\mathbf{R}^3} B_{s,\sigma,k}(x,\varepsilon),$$

where

$$B_{s,\sigma,k}(x,\varepsilon) = \langle x\rangle^{-2\sigma-2k+2}\int_{\mathbf{R}^3}|x-y|^{2k}e^{-2\varepsilon|x-y|}\langle y\rangle^{-2s-2k-2}dy$$

$$= \langle x\rangle^{-2\sigma-2k+2}\int_{|y|\le|x|/2} + \langle x\rangle^{-2\sigma-2k+2}\int_{|y|\ge|x|/2}$$

$$\le C\langle x\rangle^{-2\sigma-2k+2}|x|^{2k}e^{-\varepsilon|x|}\int_{|y|\le|x|/2}\langle y\rangle^{-2s-2k-2}dy + C\int_{\mathbf{R}^3} e^{-2\varepsilon|z|}\langle z\rangle^{-2s-2}dz$$

$$\le C\varepsilon^{2\sigma-2} + C_\epsilon\varepsilon^{-2\max\{1/2-s+\epsilon,0\}},$$

$\forall 0<\epsilon\ll 1$, uniformly in x. The estimate (2.9) is obtained in precisely the same way. □

Define the perturbed resolvent by

$$R(\lambda \pm i\varepsilon) = (G - (\lambda \pm i\varepsilon)^2)^{-1} : L^2 \to L^2,$$

and denote $R^{(k)}(\lambda \pm i\varepsilon) := d^k R(\lambda \pm i\varepsilon)/d\lambda^k$, $k \geq 1$. Let $k_0 \geq 0$ be the bigest integer strictly less than δ_0, and set $\delta_0' = \delta_0 - k_0 > 0$.

Proposition 2.2 *Under the assumption (1.1), for every $\lambda_0 > 0$, $s > -1/2$, $s_1 > 1/2$, $\lambda \geq \lambda_0$, $0 < \varepsilon \leq 1$, the following estimates hold:*

$$\|\langle x\rangle^{-s} R(\lambda \pm i\varepsilon)\langle x\rangle^{-s_1}\|_{L^2 \to L^2} \leq C_\epsilon \lambda^{-1} \varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \tag{2.11}$$

$$\left\|\langle x\rangle^{-k-s} R^{(k)}(\lambda \pm i\varepsilon)\langle x\rangle^{-k-s_1}\right\|_{L^2 \to L^2} \leq C_\epsilon \lambda^{-1} \varepsilon^{-\max\{1/2-s+\epsilon,0\}}, \quad k = 1, ..., k_0 + 1, \tag{2.12}$$

$$\left\|\langle x\rangle^{-k_0-2-s} R^{(k_0+2)}(\lambda \pm i\varepsilon)\langle x\rangle^{-k_0-2-s}\right\|_{L^2 \to L^2} \leq C_\epsilon \lambda^{-1} \varepsilon^{-1+\min\{s+1/2-\epsilon,\delta_0'\}}, \tag{2.13}$$

$\forall\, 0 < \epsilon \ll 1$, where the constant $C_\epsilon > 0$ may depend also on λ_0.

Proof. Clearly, it suffices to prove (2.11) and (2.12) for $1/2 < s_1 \leq (1 + \delta_0')/2$. We are going to take advantage of the identity

$$\langle x\rangle^{-j-s} R(\lambda \pm i\varepsilon)\langle x\rangle^{-j-s_1} (1 + K_j(\lambda \pm i\varepsilon)) = \langle x\rangle^{-j-s} R_0(\lambda \pm i\varepsilon)\langle x\rangle^{-j-s_1}, \tag{2.14}$$

where $0 \leq j \leq k_0 + 1$, and the operator

$$K_j(\lambda \pm i\varepsilon) = \langle x\rangle^{j+s_1} V R_0(\lambda \pm i\varepsilon)\langle x\rangle^{-j-s_1}$$

takes values in the compact operators in $\mathcal{L}(L^2)$. By (2.1), we have

$$\|K_j(\lambda \pm i\varepsilon)\|_{L^2 \to L^2} \leq C\lambda^{-1} \leq 1/2, \tag{2.15}$$

for $\lambda \geq \lambda_0$, $0 \leq \varepsilon \leq 1$, with some $\lambda_0 > 0$. Hence, for these values of λ we have

$$\|\,(1 + K_j(\lambda \pm i\varepsilon))^{-1}\,\|_{L^2 \to L^2} \leq C, \tag{2.16}$$

with a constant $C > 0$ independent of λ and ε. Moreover, since the operator G has no strictly positive real resonances, it is easy to see that in fact (2.16) holds for every $\lambda_0 > 0$ with a constant $C > 0$ depending on λ_0. Then (2.11) follows from (2.2), (2.14) and (2.16).

Differentiating (2.14) j times, we get

$$\langle x\rangle^{-j-s} R^{(j)}(\lambda \pm i\varepsilon)\langle x\rangle^{-j-s_1} (1 + K_j(\lambda \pm i\varepsilon)) = \langle x\rangle^{-j-s} R_0^{(j)}(\lambda \pm i\varepsilon)\langle x\rangle^{-j-s_1}$$

$$-\sum_{\nu=0}^{j-1} \beta_{\nu,j} \langle x\rangle^{-j-s} R^{(\nu)}(\lambda \pm i\varepsilon)\langle x\rangle^{-\nu-s_1} V \langle x\rangle^{\nu+s_1} R_0^{(j-\nu)}(\lambda \pm i\varepsilon)\langle x\rangle^{-j-s_1}. \tag{2.17}$$

Now it is easy to see that (2.12) follows by induction from (2.16) and (2.17) combined with (2.3) and (2.11).

To prove (2.13) we will proceed in a way similar to that one in [3]. Denote by $\widetilde{G}$ the self-adjoint realization of the operator $-\widetilde{\Delta} + V$ on the Hilbert space H introduced above. Clearly, $\widetilde{G}$ is unitary equivalent to G, so it suffices to prove (2.13) with G and L^2 replaced by $\widetilde{G}$ and H, respectively. Using (2.10) we obtain the following representation for the first derivative of the resolvent

$$\widetilde{R}(\lambda \pm i\varepsilon) = (\widetilde{G} - (\lambda \pm i\varepsilon)^2)^{-1} : H \to H,$$

namely

$$\begin{aligned}
(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} &= -\langle r\rangle^{-2-k_0-s}\widetilde{R}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
+\langle r\rangle^{-2-k_0-s}\widetilde{R}(\lambda \pm i\varepsilon)\partial_r r\langle r\rangle^{-2-k_0-s} &- \langle r\rangle^{-2-k_0-s} r\partial_r \widetilde{R}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
&+\langle r\rangle^{-2-k_0-s}\widetilde{R}(\lambda \pm i\varepsilon)\partial_r rV\widetilde{R}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
&-\langle r\rangle^{-2-k_0-s}\widetilde{R}(\lambda \pm i\varepsilon)Vr\partial_r\widetilde{R}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
&+\langle r\rangle^{-2-k_0-s}\widetilde{R}(\lambda \pm i\varepsilon)V\widetilde{R}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}. \qquad (2.18)
\end{aligned}$$

Differentiating (2.18) $k_0 + 1$ times with respect to λ leads to

$$\begin{aligned}
(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+2)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} &= -2\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
+\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1)}(\lambda \pm i\varepsilon)\partial_r r\langle r\rangle^{-2-k_0-s} &- \langle r\rangle^{-2-k_0-s} r\partial_r\widetilde{R}^{(k_0+1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
&+\sum_{\nu=0}^{k_0+1}\alpha_\nu\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1-\nu)}(\lambda \pm i\varepsilon)\partial_r rV\widetilde{R}^{(\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
&-\sum_{\nu=0}^{k_0+1}\alpha_\nu\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1-\nu)}(\lambda \pm i\varepsilon)Vr\partial_r\widetilde{R}^{(\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s} \\
&+\sum_{\nu=0}^{k_0+1}\alpha_\nu\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1-\nu)}(\lambda \pm i\varepsilon)V\widetilde{R}^{(\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}. \qquad (2.19)
\end{aligned}$$

By (2.19), we obtain

$$\begin{aligned}
|\lambda \pm i\varepsilon|\left\|\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+2)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| &\le C\left\|\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| \\
+C\left\|b_{s+k_0+1}(r)\partial_r\widetilde{R}^{(k_0+1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| &+ C\left\|\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1)}(\lambda \pm i\varepsilon)\partial_r b_{s+k_0+1}(r)\right\| \\
+C\sum_{\nu=0}^{k_0+1}\left\|\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(k_0+1-\nu)}(\lambda \pm i\varepsilon)\partial_r b_{s_0+k_0+1-\nu}(r)\right\| &\left\|\langle r\rangle^{-s_0-\nu}\widetilde{R}^{(\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| \\
+C\sum_{\nu=0}^{k_0+1}\left\|\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-s_0-\nu}\right\| &\left\|b_{s_0+k_0+1-\nu}(r)\partial_r\widetilde{R}^{(k_0+1-\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| \\
+C\sum_{\nu=0}^{k_0+1}\left\|\langle r\rangle^{-2-k_0-s}\widetilde{R}^{(\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-s_0-\nu}\right\| &\left\|\langle r\rangle^{-s_0-k_0-1+\nu}\widetilde{R}^{(k_0+1-\nu)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\|, \qquad (2.20)
\end{aligned}$$

where $b_s(r) = r\langle r\rangle^{-1-s}$, $s_0 = \delta_0'/2$, and $\|\cdot\|$ denotes the norm on $\mathcal{L}(H)$.

Given any $f \in H$, the function $u = \widetilde{R}(\lambda \pm i\varepsilon)f$ satisfies the equation

$$\left(-\partial_r^2 + r^{-2}\Delta_{S^2} + V - (\lambda \pm i\varepsilon)^2\right)u = f,$$

so we have

$$\left(-\partial_r^2 + r^{-2}\Delta_{S^2} + V - (\lambda \pm i\varepsilon)^2\right) b_s(r)u = b_s(r)f + [-\partial_r^2, b_s(r)]u.$$

Integrating by parts yields, $\forall \gamma > 0$,

$$\begin{aligned}
\|\partial_r(b_s(r)u)\|_H^2 &\le \left(C_1 + |\lambda \pm i\varepsilon|^2\right) \left\|\langle r\rangle^{-s}u\right\|_H^2 \\
&\quad + \left|\left\langle b_s(r)f + [-\partial_r^2, b_s(r)]u, b_s(r)u\right\rangle_H\right| \\
&\le \left(C_2 + |\lambda \pm i\varepsilon|^2\right) \left\|\langle r\rangle^{-s}u\right\|_H^2 + C_3\|b_s(r)f\|_H^2 + \gamma^2 \left\|r\langle r\rangle^{-1}[-\partial_r^2, b_s(r)]u\right\|_H^2 \\
&\le \left(C_4 + |\lambda \pm i\varepsilon|^2\right) \left\|\langle r\rangle^{-s}u\right\|_H^2 + C_3\|b_s(r)f\|_H^2 + O(\gamma^2)\,\|b_s(r)\partial_r u\|_H^2 .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|b_s(r)\partial_r u\|_H &\le \|\partial_r\left(b_s(r)u\right)\|_H + C\left\|\langle r\rangle^{-s}u\right\|_H \\
&\le (C + |\lambda \pm i\varepsilon|)\left\|\langle r\rangle^{-s}u\right\|_H + C\|b_s(r)f\|_H + O(\gamma)\,\|b_s(r)\partial_r u\|_H ,
\end{aligned}$$

which, after taking γ small enough, gives

$$\|b_s(r)\partial_r u\|_H \le C\left(1 + |\lambda \pm i\varepsilon|\right)\left\|\langle r\rangle^{-s}u\right\|_H + C\|b_s(r)f\|_H. \tag{2.21}$$

By (2.21) we get, for $j = 0, 1, ...,$

$$\begin{aligned}
\left\|b_{s+j}(r)\partial_r \widetilde{R}^{(j)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| &\le C(1 + |\lambda \pm i\varepsilon|)\left\|\langle r\rangle^{-j-s}\widetilde{R}^{(j)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\| \\
&\quad + C|\lambda \pm i\varepsilon|\left\|\langle r\rangle^{-j-s}\widetilde{R}^{(j-1)}(\lambda \pm i\varepsilon)\langle r\rangle^{-2-k_0-s}\right\|,
\end{aligned} \tag{2.22}$$

where $(\lambda \pm i\varepsilon)\widetilde{R}^{(-1)}(\lambda \pm i\varepsilon) := Id$. By (2.20) and (2.22) combined with (2.11) and (2.12) we obtain

$$|\lambda \pm i\varepsilon|\left\|\langle x\rangle^{-2-k_0-s}R^{(k_0+2)}(\lambda \pm i\varepsilon)\langle x\rangle^{-2-k_0-s}\right\|_{L^2\to L^2} \le C_\epsilon \varepsilon^{-1+\min\{1/2+s-\epsilon,1\}} + C_\epsilon \varepsilon^{-1+\delta_0'-\epsilon},$$

which clearly implies (2.13). $\square$

3 Time decay estimates

Given parameters $A \gg a > 0$, choose a function $\psi_{a,A} \in C_0^\infty([a, A])$ such that

$$\left|\partial_y^k \psi_{a,A}(y)\right| \le C_k, \quad \forall k \ge 0,$$

with a constant $C_k > 0$ independent of A. For $z \in \mathbf{C}$, set

$$\Phi_{A,z}(t) = \langle x\rangle^{-\sigma z} G^{-z} e^{it\sqrt{G}} \psi_{a,A}(\sqrt{G})\langle x\rangle^{-\sigma z} - \langle x\rangle^{-\sigma z} G_0^{-z} e^{it\sqrt{G_0}} \psi_{a,A}(\sqrt{G_0})\langle x\rangle^{-\sigma z}.$$

Theorem 3.1 *Under the assumption (1.1), for every $a > 0$, $2 \le p \le +\infty$, $A \gg a$, $0 < \sigma < \delta_0$, we have*

$$\|\Phi_{A,\alpha}(t)\|_{L^{p'}\to L^p} \le C\left(|t|^{-1-\sigma}\log A\right)^\alpha, \quad |t| \ge 1, \tag{3.1}$$

with a constant $C > 0$ independent of t and A, where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$.

Proof. We will first prove (3.1) for $p=+\infty$, $p'=1$. We have

$$\Phi_{A,z}(t)=\int_0^\infty e^{it\lambda}\lambda^{1-2z}\psi_{a,A}(\lambda)T(\lambda;\sigma z)d\lambda, \tag{3.2}$$

where $z\in\mathbf{C}$, $\operatorname{Re} z=1$, and

$$T(\lambda;\sigma z)=(\pi i)^{-1}\left(T^+(\lambda;\sigma z)-T^-(\lambda;\sigma z)\right),$$

$$T^\pm(\lambda;\sigma z)=\langle x\rangle^{-\sigma z}\left(R(\lambda\pm i0)-R_0(\lambda\pm i0)\right)\langle x\rangle^{-\sigma z}$$

$$=\langle x\rangle^{-\sigma z}R_0(\lambda\pm i0)VR_0(\lambda\pm i0)\langle x\rangle^{-\sigma z}+\langle x\rangle^{-\sigma z}R_0(\lambda\pm i0)VR(\lambda\pm i0)VR_0(\lambda\pm i0)\langle x\rangle^{-\sigma z}.$$

Denote by $j_0\ge 0$ the bigest integer strictly less than σ, and denote $\sigma'=\sigma-j_0>0$.

Proposition 3.2 *Under the assumption (1.1), if* $0<\sigma<\delta_0$ *the operator-valued functions* $T^\pm(\lambda;\sigma z):L^1\to L^\infty$ *satisfy the estimates*

$$\|\partial_\lambda^j T^\pm(\lambda;\sigma z)\|_{L^1\to L^\infty}\le C,\quad \lambda\ge\lambda_0,\quad j=0,1,...,j_0+1, \tag{3.3}$$

$$\|\partial_\lambda^{j_0+1}T^\pm(\lambda_2;\sigma z)-\partial_\lambda^{j_0+1}T^\pm(\lambda_1;\sigma z)\|_{L^1\to L^\infty}\le C|\lambda_2-\lambda_1|^{\sigma'},\quad \lambda_2,\lambda_1\ge\lambda_0, \tag{3.4}$$

$\forall\,\lambda_0>0$ *with a constant* $C>0$ *which may depend on* λ_0 *but is independent of* λ, λ_1, λ_2 *and* z. *If* $\sigma=\delta_0$ *we have (3.3) and (3.4) with* $j_0=k_0$ *and* $\forall\,0<\sigma'<\delta_0'$.

Proof. Let first $0<\sigma<\delta_0$. This implies $j_0\le k_0$. For every integer $j\ge 0$ we have

$$\partial_\lambda^j T^\pm(\lambda;\sigma z)$$

$$=\sum_{\nu_1+\nu_2+\nu_3=j}\alpha_{\nu_1,\nu_2,\nu_3}\langle x\rangle^{-\sigma z}\partial_\lambda^{\nu_1}R_0(\lambda\pm i0)V\partial_\lambda^{\nu_2}R(\lambda\pm i0)V\partial_\lambda^{\nu_3}R_0(\lambda\pm i0)\langle x\rangle^{-\sigma z}$$

$$+\sum_{\mu_1+\mu_2=j}\beta_{\mu_1,\mu_2}\langle x\rangle^{-\sigma z}\partial_\lambda^{\mu_1}R_0(\lambda\pm i0)V\partial_\lambda^{\mu_2}R_0(\lambda\pm i0)\langle x\rangle^{-\sigma z}$$

$$:=\sum_{\nu_1+\nu_2+\nu_3=j}\mathcal{A}_{\nu_1,\nu_2,\nu_3}(\lambda;\varepsilon)+\sum_{\mu_1+\mu_2=j}\mathcal{B}_{\mu_1,\mu_2}(\lambda;\varepsilon). \tag{3.5}$$

Let $j\le j_0+1$. Applying (2.4), (2.5), (2.7), (2.8) with $s>1/2$, (2.6) with $k=\nu_1-1$ (when $\nu_1\ge 2$), (2.7) with $k=\nu_3-1$ (when $\nu_3\ge 2$), and (2.11), (2.12) with $k=\nu_2$, $s>1/2$, we get

$$\|\mathcal{A}_{\nu_1,\nu_2,\nu_3}(\lambda;\varepsilon)\|_{L^1\to L^\infty}\le C, \tag{3.6}$$

with a constant $C>0$ independent of λ and ε. Clearly, a similar estimate holds for $\mathcal{B}_{\mu_1,\mu_2}(\lambda;\varepsilon)$, and hence (3.3) follows.

To prove (3.4) it suffices to show that

$$\|\partial_\lambda^{j_0+2}T^\pm(\lambda\pm i\varepsilon;\sigma z)\|_{L^1\to L^\infty}\le C\varepsilon^{-1+\sigma'}. \tag{3.7}$$

Indeed, (3.7) implies, $\forall 0<\varepsilon\le 1$,

$$\|\partial_\lambda^{j_0+1}T^\pm(\lambda_2\pm i\varepsilon;\sigma z)-\partial_\lambda^{j_0+1}T^\pm(\lambda_1\pm i\varepsilon;\sigma z)\|_{L^1\to L^\infty}\le C|\lambda_2-\lambda_1|\varepsilon^{-1+\sigma'},$$

$$\|\partial_\lambda^{j_0+1}T^\pm(\lambda_k\pm i\varepsilon;\sigma z)-\partial_\lambda^{j_0+1}T^\pm(\lambda_k;\sigma z)\|_{L^1\to L^\infty}\le C\varepsilon^{\sigma'},\quad k=1,2,$$

which yield, $\forall 0 < \varepsilon \leq 1$,

$$\|\partial_\lambda^{j_0+1} T^\pm(\lambda_2; \sigma z) - \partial_\lambda^{j_0+1} T^\pm(\lambda_1; \sigma z)\|_{L^1 \to L^\infty} \leq C\varepsilon^{\sigma'} \left(2 + |\lambda_2 - \lambda_1|\varepsilon^{-1}\right). \tag{3.8}$$

Thus, (3.4) follows from (3.8) by taking $\varepsilon = |\lambda_2 - \lambda_1|$.

To prove (3.7) we will make use of (3.5) with $j = j_0 + 2$. If $j_0 < k_0$ we have $j \leq k_0 + 1$ and hence this case can be treated in the same way as above to get (3.7) with $\sigma' = 1$. Let now $j_0 = k_0$. Then, since $\sigma < \delta_0$, we have $\sigma' < \delta_0'$. Using (2.4), (2.7) with $s = 1/2 + \epsilon$, $0 < \epsilon \ll 1$, and (2.13) with $s = \delta_0' - 1/2 - \epsilon$, we obtain

$$\|\mathcal{A}_{0,j_0+2,0}(\lambda; \varepsilon)\|_{L^1 \to L^\infty} \leq C \left\|R_0(\lambda \pm i\varepsilon)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2 \to L^\infty}$$

$$\left\|\langle x\rangle^{-2-k_0-\delta_0'+1/2+\epsilon} R^{(k_0+2)}(\lambda \pm i\varepsilon)\langle x\rangle^{-2-k_0-\delta_0'+1/2+\epsilon}\right\|_{L^2 \to L^2} \left\|\langle x\rangle^{-1/2-\epsilon} R_0(\lambda \pm i\varepsilon)\right\|_{L^1 \to L^2}$$

$$\leq C_\epsilon \varepsilon^{-1+\delta_0'-2\epsilon} \leq C\varepsilon^{-1+\sigma'}, \tag{3.9}$$

provided $\epsilon > 0$ is taken small enough. Using (2.5), (2.7) with $s = 1/2 + \epsilon$, and (2.12) with $k = k_0 + 1$, $s_1 = 1/2 + \delta_0' - \epsilon$, $s = \delta_0' - 1/2 - \epsilon$, we obtain

$$\|\mathcal{A}_{1,j_0+1,0}(\lambda; \varepsilon)\|_{L^1 \to L^\infty} \leq C \left\|R_0^{(1)}(\lambda \pm i\varepsilon)\langle x\rangle^{-3/2-\epsilon}\right\|_{L^2 \to L^\infty}$$

$$\left\|\langle x\rangle^{-1-k_0-\delta_0'+1/2+\epsilon} R^{(k_0+1)}(\lambda \pm i\varepsilon)\langle x\rangle^{-2-k_0-\delta_0'+1/2+\epsilon}\right\|_{L^2 \to L^2} \left\|\langle x\rangle^{-1/2-\epsilon} R_0(\lambda \pm i\varepsilon)\right\|_{L^1 \to L^2}$$

$$\leq C_\epsilon \varepsilon^{-1+\delta_0'-2\epsilon} \leq C\varepsilon^{-1+\sigma'}, \tag{3.10}$$

and similarly for $\mathcal{A}_{0,j_0+1,1}(\lambda; \varepsilon)$. Let now $\nu_1 = j_0 + 2$. Using (2.6) with $k = j_0 + 1$, $\sigma = \sigma'$, $s = \delta_0' - 1/2 - \epsilon$, (2.7) with $s = 1/2 + \epsilon$, and (2.11) with $s = s_1 = 1/2 + \epsilon$, we obtain

$$\|\mathcal{A}_{j_0+2,0,0}(\lambda; \varepsilon)\|_{L^1 \to L^\infty} \leq C \left\|\langle x\rangle^{-j_0-\sigma'} R_0^{(j_0+2)}(\lambda \pm i\varepsilon)\langle x\rangle^{-j_0-2-\delta_0'+1/2+\epsilon}\right\|_{L^2 \to L^\infty}$$

$$\left\|\langle x\rangle^{-1/2-\epsilon} R(\lambda \pm i\varepsilon)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2 \to L^2} \left\|\langle x\rangle^{-1/2-\epsilon} R_0(\lambda \pm i\varepsilon)\right\|_{L^1 \to L^2}$$

$$\leq C\varepsilon^{-1+\sigma'}, \tag{3.11}$$

and similarly for $\mathcal{A}_{0,0,j_0+2}(\lambda; \varepsilon)$. Let now $\nu_2 \leq j_0$, $\nu_1 \leq j_0 + 1$, $\nu_3 \leq j_0 + 1$. This implies $\nu_1 + \nu_3 \geq 2$, $\nu_2 + \nu_3 \geq 1$, $\nu_2 + \nu_1 \geq 1$. As above we have

$$\|\mathcal{A}_{\nu_1,\nu_2,\nu_3}(\lambda; \varepsilon)\|_{L^1 \to L^\infty} \leq C \left\|\langle x\rangle^{-\nu_1+1-\sigma'} R_0^{(\nu_1)}(\lambda \pm i\varepsilon)\langle x\rangle^{-\nu_1-\nu_3-\delta_0'+1/2+\epsilon}\right\|_{L^2 \to L^\infty}$$

$$\left\|\langle x\rangle^{-\nu_2-1/2-\epsilon} R^{(\nu_2)}(\lambda \pm i\varepsilon)\langle x\rangle^{-\nu_2-1/2-\epsilon}\right\|_{L^2 \to L^2}$$

$$\left\|\langle x\rangle^{-\nu_1-\nu_3-\delta_0'+1/2+\epsilon} R_0^{(\nu_3)}(\lambda \pm i\varepsilon)\langle x\rangle^{-\nu_3+1-\sigma'}\right\|_{L^1 \to L^2}$$

$$\leq C_\epsilon \varepsilon^{-1+\delta_0'-2\epsilon} \leq C\varepsilon^{-1+\sigma'}. \tag{3.12}$$

It follows from (3.9)-(3.12) that the first sum in the RHS of (3.5) is $O\left(\varepsilon^{-1+\sigma'}\right)$. In the same way it is easy to see that the second sum satisfies the same bound, and hence (3.7) follows. When $\sigma = \delta_0$, as above one can show that (3.7) holds with any $0 < \sigma' < \delta_0'$. $\square$

We will first consider the case of $0 < \sigma < \delta_0$. Let $\phi \in C_0^\infty([1/3, 1/2])$ be a real-valued function, $\phi \geq 0$, such that $\int \phi(y)dy = 1$. Then, the function

$$T_\epsilon^\pm(\lambda; \sigma z) = \epsilon^{-1} \int T^\pm(\lambda - y; \sigma z)\phi(y/\epsilon)dy, \quad 0 < \epsilon \ll 1,$$

is smooth with values in $\mathcal{L}(L^1, L^\infty)$ and, in view of (3.3) and (3.4), satisfies the estimates

$$\|\partial_\lambda^j T_\epsilon^\pm(\lambda; \sigma z)\|_{L^1 \to L^\infty} \leq C, \quad j = 0, 1, ..., j_0 + 1, \tag{3.13}$$

$$\begin{aligned} &\|\partial_\lambda^{j_0+1} T_\epsilon^\pm(\lambda; \sigma z) - \partial_\lambda^{j_0+1} T^\pm(\lambda; \sigma z)\|_{L^1 \to L^\infty} \\ &\leq \epsilon^{-1} \int \|\partial_\lambda^{j_0+1} T^\pm(\lambda; \sigma z) - \partial_\lambda^{j_0+1} T^\pm(\lambda - y; \sigma z)\|_{L^1 \to L^\infty} \phi(y/\epsilon)dy \\ &\leq C\epsilon^{-1} \int y^{\sigma'} \phi(y/\epsilon)dy = O(\epsilon^{\sigma'}). \end{aligned} \tag{3.14}$$

Let us see that we also have

$$\|\partial_\lambda^{j_0+2} T_\epsilon^\pm(\lambda; \sigma z)\|_{L^1 \to L^\infty} \leq C\epsilon^{-1+\sigma'}. \tag{3.15}$$

Given any $0 \leq \varepsilon \leq 1$, define

$$T_\epsilon^\pm(\lambda \pm i\varepsilon; \sigma z) = \epsilon^{-1} \int T^\pm(\lambda \pm i\varepsilon - y; \sigma z)\phi(y/\epsilon)dy.$$

In view of (3.7), we have

$$\|\partial_\lambda^{j_0+2} T_\epsilon^\pm(\lambda \pm i\varepsilon; \sigma z)\|_{L^1 \to L^\infty} \leq C\varepsilon^{-1+\sigma'}, \tag{3.16}$$

with a constant $C > 0$ independent of λ, ε and ϵ. On the other hand,

$$\begin{aligned} &\|\partial_\lambda^{j_0+2} T_\epsilon^\pm(\lambda \pm i\varepsilon; \sigma z) - \partial_\lambda^{j_0+2} T_\epsilon^\pm(\lambda; \sigma z)\|_{L^1 \to L^\infty} \\ &\leq \epsilon^{-2} \int \|\partial_\lambda^{j_0+1} T_\epsilon^\pm(\lambda - y \pm i\varepsilon; \sigma z) - \partial_\lambda^{j_0+1} T_\epsilon^\pm(\lambda - y; \sigma z)\|_{L^1 \to L^\infty} |\phi'(y/\epsilon)|dy \\ &\leq C\varepsilon^{\sigma'}\epsilon^{-2} \int |\phi'(y/\epsilon)|dy \leq C\varepsilon^{\sigma'}\epsilon^{-1}. \end{aligned} \tag{3.17}$$

By (3.16) and (3.17),

$$\|\partial_\lambda^{j_0+2} T_\epsilon^\pm(\lambda; \sigma z)\|_{L^1 \to L^\infty} \leq C\varepsilon^{\sigma'}(\varepsilon^{-1} + \epsilon^{-1}),$$

which implies (3.15) if we take $\varepsilon = \epsilon$.

Integrating by parts we obtain

$$\begin{aligned} (it)^{j_0+1}\Phi_{A,z}(t) &= \int_0^\infty e^{it\lambda} \frac{d^{j_0+1}}{d\lambda^{j_0+1}} \left(\lambda^{1-2z}\psi_{a,A}(\lambda)T(\lambda; \sigma z)\right) d\lambda \\ &= \sum_{\nu=0}^{j_0+1} \gamma_\nu \int_0^\infty e^{it\lambda} \frac{d^{j_0+1-\nu}}{d\lambda^{j_0+1-\nu}} \left(\lambda^{1-2z}\psi_{a,A}(\lambda)\right) \partial_\lambda^\nu T(\lambda; \sigma z)d\lambda \end{aligned}$$

$$= (it)^{-1}\sum_{\nu=0}^{j_0}\gamma_\nu\int_0^\infty e^{it\lambda}\frac{d}{d\lambda}\left(\frac{d^{j_0+1-\nu}}{d\lambda^{j_0+1-\nu}}\left(\lambda^{1-2z}\psi_{a,A}(\lambda)\right)\partial_\lambda^\nu T(\lambda;\sigma z)\right)d\lambda$$

$$+\int_0^\infty e^{it\lambda}\lambda^{1-2z}\psi_{a,A}(\lambda)\partial_\lambda^{j_0+1}T(\lambda;\sigma z)d\lambda := I_1(t)+I_2(t).$$

In view of (3.3) we have

$$\|I_1(t)\|_{L^1\to L^\infty}\le C|t|^{-1}\langle z\rangle^{j_0+2}\log A. \tag{3.18}$$

On the other hand, by (3.14) we get

$$\left\|\int_0^\infty e^{it\lambda}\lambda^{1-2z}\psi_{a,A}(\lambda)\left(\partial_\lambda^{j_0+1}T^\pm(\lambda;\sigma z)-\partial_\lambda^{j_0+1}T^\pm_\epsilon(\lambda;\sigma z)\right)d\lambda\right\|_{L^1\to L^\infty}$$

$$\le C\epsilon^{\sigma'}\int\lambda^{-1}|\psi_{a,A}(\lambda)|d\lambda\le C\epsilon^{\sigma'}\log A. \tag{3.19}$$

By (3.15) we get

$$\left\|\int_0^\infty e^{it\lambda}\lambda^{1-2z}\psi_{a,A}(\lambda)\partial_\lambda^{j_0+1}T^\pm_\epsilon(\lambda;\sigma z)d\lambda\right\|_{L^1\to L^\infty}$$

$$=\left\|t^{-1}\int_0^\infty e^{it\lambda}\frac{d}{d\lambda}\left(\lambda^{1-2z}\psi_{a,A}(\lambda)\partial_\lambda^{j_0+1}T^\pm_\epsilon(\lambda;\sigma z)\right)d\lambda\right\|_{L^1\to L^\infty}\le C|t|^{-1}\epsilon^{-1+\sigma'}\langle z\rangle\log A. \tag{3.20}$$

Taking $\epsilon=|t|^{-1}$ we deduce from (3.19) and (3.20),

$$\|I_2(t)\|_{L^1\to L^\infty}\le C|t|^{-\sigma'}\langle z\rangle\log A. \tag{3.21}$$

By (3.18) and (3.21),

$$\|\Phi_{A,z}(t)\|_{L^1\to L^\infty}\le C|t|^{-1-\sigma}\langle z\rangle^{j_0+2}\log A, \tag{3.22}$$

$\forall z\in\mathbf{C}$, $\operatorname{Re} z=1$. On the other hand, we have the trivial estimate

$$\|\Phi_{A,z}(t)\|_{L^2\to L^2}\le C, \tag{3.23}$$

$\forall z\in\mathbf{C}$, $\operatorname{Re} z=0$. Now (3.1) follows from (3.22) and (3.23) by analytic interpolation. □

Let $\varphi\in C_0^\infty([1/2,2])$, independent of the parameter A. For $z\in\mathbf{C}$, set

$$F_{A,z}(t)=\langle x\rangle^{-\sigma z}G^{-z}e^{it\sqrt{G}}\varphi(\sqrt{G}/A)\langle x\rangle^{-\sigma z}-\langle x\rangle^{-\sigma z}G_0^{-z}e^{it\sqrt{G_0}}\varphi(\sqrt{G_0}/A)\langle x\rangle^{-\sigma z}.$$

The following proposition is proved in [3] for potentials $V(x)=O\left(\langle x\rangle^{-1-\varepsilon_0}\right)$, $\varepsilon_0>0$.

Proposition 3.3 *For every $2\le p\le+\infty$, $|t|\ge1$, $A\gg1$, we have*

$$\|F_{A,\alpha}(t)\|_{L^{p'}\to L^p}\le C|t|^{2/p}A^{-2/p}, \tag{3.24}$$

with a constant $C>0$ independent of t and A, where $1/p+1/p'=1$, $\alpha=1-2/p$.

We are going to show now that Theorem 3.1 together with Proposition 3.3 imply Theorem 1.1. Choose a function $\varphi\in C_0^\infty([1/2,1])$ such that $\int\varphi(y)dy=1$, and denote $\varphi_1(y)=y\varphi(y)$. For $A\gg1$ we can write

$$\chi_a(y)=\psi_{A,a}(y)+\eta_A(y), \tag{3.25}$$

where

$$\psi_{A,a}(y)=\chi_a(y)A^{-1}\int_y^{+\infty}\varphi(\tau/A)d\tau,$$

$$\eta_A(y)=A^{-1}\int_0^y\varphi(\tau/A)d\tau=\int_0^1\varphi_1(sy/A)s^{-1}ds.$$

In view of (3.24), since $p<+\infty$, we have

$$\begin{aligned}&\left\|\langle x\rangle^{-\sigma\alpha}G^{-\alpha}e^{it\sqrt{G}}\eta_A(\sqrt{G})\langle x\rangle^{-\sigma\alpha}-\langle x\rangle^{-\sigma\alpha}G_0^{-\alpha}e^{it\sqrt{G_0}}\eta_A(\sqrt{G_0})\langle x\rangle^{-\sigma\alpha}\right\|_{L^{p'}\to L^p}\\ &=\left\|\int_0^1\left(\langle x\rangle^{-\sigma\alpha}G^{-\alpha}e^{it\sqrt{G}}\varphi_1(s\sqrt{G}/A)\langle x\rangle^{-\sigma\alpha}\right.\right.\\ &\left.\left.-\langle x\rangle^{-\sigma\alpha}G_0^{-\alpha}e^{it\sqrt{G_0}}\varphi_1(s\sqrt{G_0}/A)\langle x\rangle^{-\sigma\alpha}\right)s^{-1}ds\right\|_{L^{p'}\to L^p}\\ &\le C|t|^{2/p}A^{-2/p}\int_0^1 s^{2/p-1}ds\le C'|t|^{2/p}A^{-2/p}.\end{aligned}\tag{3.26}$$

Combining (3.25), (3.26) and (3.1) we get

$$\begin{aligned}&\left\|\langle x\rangle^{-\sigma\alpha}G^{-\alpha}e^{it\sqrt{G}}\chi_a(\sqrt{G})\langle x\rangle^{-\sigma\alpha}-\langle x\rangle^{-\sigma\alpha}G_0^{-\alpha}e^{it\sqrt{G_0}}\chi_a(\sqrt{G_0})\langle x\rangle^{-\sigma\alpha}\right\|_{L^{p'}\to L^p}\\ &\le C\left(|t|^{-1-\sigma}\log A\right)^\alpha+C'|t|^{2/p}A^{-2/p},\end{aligned}\tag{3.27}$$

for every $A\gg1$. Now (1.4) follows from (3.27) by taking $A=(|t|+1)^k$ with $k>0$ big enough, together with (1.3). Note finally that (1.5) follows in the same way by observing that when $\sigma=\delta_0$ (3.21) holds for every $0<\sigma'<\delta_0'$. □

Appendix

In what follows we will derive (1.3) from (1.2). Denote by $\eta(x,t)$ the characteristic function of the set $\{|x|\le|t|/4\}$. Indeed, it is easy to see that (1.3) follows from (1.2), the fact that the operator $\chi_a(\sqrt{G_0})$ is bounded on L^p, $2\le p<+\infty$, and the following estimate

$$\left\|\eta G_0^{-\alpha}e^{it\sqrt{G_0}}\chi_a(\sqrt{G_0})\eta\right\|_{L^{p'}\to L^p}\le C_N|t|^{-\alpha N},\quad |t|\ge1,\tag{A.1}$$

for every integer $N\ge1$ and every $2\le p\le+\infty$, where $1/p+1/p'=1$, $\alpha=1-2/p$. To prove (A.1) observe that the kernel, $K_z(x,y;t)$, of the operator $\eta G_0^{-z}e^{it\sqrt{G_0}}\chi_a(\sqrt{G_0})\eta$, $z\in\mathbf{C},\operatorname{Re}z=1$, is given by the oscilatory integral

$$\begin{aligned}K_z(x,y;t)&=\eta(x,t)\eta(y,t)(2\pi)^{-3}\int_{\mathbf{R}^3}e^{it|\xi|-i\langle x-y,\xi\rangle}|\xi|^{-2z}\chi_a(|\xi|)d\xi\\ &=\eta(x,t)\eta(y,t)(2\pi)^{-3}\int_{\mathbf{S}^2}\int_0^\infty e^{i\rho(t-\langle x-y,w\rangle)}\rho^{2-2z}\chi_a(\rho)d\rho dw\\ &=c_N\int_{\mathbf{S}^2}\eta(x,t)\eta(y,t)(t-\langle x-y,w\rangle)^{-N}\int_0^\infty e^{i\rho(t-\langle x-y,w\rangle)}\partial_\rho^N\left(\rho^{2-2z}\chi_a(\rho)\right)d\rho dw.\end{aligned}$$

Hence,

$$|K_z(x,y;t)|\le C_N'|t|^{-N}\int_0^\infty\left|\partial_\rho^N\left(\rho^{2-2z}\chi_a(\rho)\right)\right|d\rho\le C_N\langle z\rangle^N|t|^{-N},\tag{A.2}$$

for every integer $N\ge2$. By (A.2) we obtain

$$\left\|\eta G_0^{-z}e^{it\sqrt{G_0}}\chi_a(\sqrt{G_0})\eta\right\|_{L^1\to L^\infty}\le C_N\langle z\rangle^N|t|^{-N},\quad |t|\ge1,\tag{A.3}$$

for every integer $N\ge2$, $z\in\mathbf{C},\operatorname{Re}z=1$. Now (A.1) follows from (A.3) by analytic interpolation.

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